TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 185, November 1973

THE STRONG LAW OF LARGE NUMBERS WHEN THE MEAN IS UNDEFINED

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ABSTRACT. Let $S_n = X_1 + \cdots + X_n$ where $\{X_n\}$ are i.i.d. random variables with $EX_1^{\pm} = \infty$. An integral test is given for each of the three possible alternatives $\lim(S_n/n) = +\infty$ a.s.; $\lim(S_n/n) = -\infty$ a.s.; $\limsup(S_n/n) = +\infty$ and $\liminf(S_n/n) = -\infty$ a.s. Some applications are noted.

1. Introduction. Let $\{X_n\}$ be a sequence of independent identically distributed random variables and put $S_n = X_1 + \cdots + X_n$, $n \geq 1$. It is well known that if EX_1 is defined in the sense that one or both of EX_1^+, EX_1^- ($x^+ = \max(x, 0)$, $x^- = \max(-x, 0)$) is finite then

$$P\left\{\lim_{n\to\infty}(S_n/n) = EX_1\right\} = 1. \tag{1.1}$$

If however $EX_1^+ = EX_1^- = \infty$ then EX_1 is undefined and (1.1) is meaningless. In this case Kesten [5, Corollary 3, p. 1195] has proved the following.

Theorem 1. *If $EX_1^+ = EX_1^- = \infty$ then one of the following alternatives must prevail*:

(i) $P\{\lim(S_n/n) = +\infty\} = 1$;
(ii) $P\{\lim(S_n/n) = -\infty\} = 1$;
(iii) $P\{\limsup(S_n/n) = +\infty$ *and* $\liminf(S_n/n) = -\infty\} = 1$.

In this paper we shall give a simple necessary and sufficient criterion, in the form of an integral test, for each of (i)–(iii).

2. Notation and statement of results. Let X stand for any of the random variables $\{X_i\}$ and assume $P\{X = 0\} \neq 1$. Put $F(t) = P\{X \leq t\}$ and define the following quantities:

Received by the editors September 22, 1972.

AMS (MOS) subject classifications (1970). Primary 60F15, 60F20, 60J15; Secondary 60G50, 60J30.

Key words and phrases. Independent identically distributed random variables, mean undefined, strong law of large numbers, renewal function, truncated mean function.

(1) Supported in part by National Science Foundation Grant GP 28876 A1.

$$m_-(x) = \int_{-x}^{0} F(y)\,dy = xF(-x) + \int_{-x}^{0} |y|\,dF(y),$$

$$m_+(x) = \int_0^x [1 - F(y)]\,dy = x[1 - F(x)] + \int_0^x y\,dF(y),$$

$$J_+ = J_+(X) = \int_{0+}^{\infty} \frac{x}{m_-(x)}\,dF(x),$$

$$J_- = J_-(X) = \int_{-\infty}^{0-} \frac{|x|}{m_+(|x|)}\,dF(x) = J_+(-X).$$

The integrand in J_+, J_- is bounded near $x = 0$ whenever $F(0-) \neq 0$ or $1 - F(0) \neq 0$ respectively. If $P\{X < 0\} = F(0-) = 0$ define $J_+ = EX = EX^+$ and if $P\{X > 0\} = 1 - F(0) = 0$ define $J_- = E|X| = EX^-$.

Note the following properties: as $t \to \infty$, $m_+(t) \to EX^+$, $m_-(t) \to EX^-$ and, since m_+ and m_- are nondecreasing,

$$J_+ \leq cEX^+, \qquad J_- \leq cEX^- \tag{2.1}$$

for some $c < \infty$ whether or not EX^+, EX^- are finite.

Theorem 2. (*No assumptions on* $EX_1^{\pm}$.)
(a) $J_+ = \infty$ *if and only if* $P\{\limsup(S_n/n) = +\infty\} = 1$;
(b) $J_- = \infty$ *if and only if* $P\{\liminf(S_n/n) = -\infty\} = 1$;
(c) $J_1 < J_+ = \infty$ *if and only if* $P\{\lim(S_n/n) = +\infty\} = 1$;
(d) $J_+ < J_- = \infty$ *if and only if* $P\{\lim(S_n/n) = -\infty\} = 1$.

Remark. It follows from the four alternatives presented in Theorem 2 and the Hewitt-Savage 0-1 law that if both J_+ and J_- are finite the sequence $\{S_n/n\}$ must be bounded with probability 1. But this is the case if and only if $E|X_1| < \infty$ (and then $\lim(S_n/n) = EX_1$ a.s.). From this and (2.1) we conclude

$$J_+ + J_- < \infty \quad \text{if and only if} \quad E|X_1| < \infty.$$

This is a purely analytic fact. For a direct analytic proof that $J_+ + J_- < \infty$ implies $E|X_1| < \infty$, see note 7 below.

Corollary 1. *Assume* $E|X_1| = \infty$. *Then at most one of* J_+, J_- *is finite and*
(a) $P\{\lim(S_n/n) = +\infty\} = 1$ *iff* $J_- < \infty$;
(b) $n\{\lim(S_n/n) = -\infty\} = 1$ *iff* $J_+ < \infty$;
(c) $P\{\underline{\lim}(S_n/n) = -\infty$ *and* $\overline{\lim}(S_n/n) = +\infty\} = 1$ *iff* $J_+ = J_- = \infty$.

Proof. This corollary follows immediately from Theorem 2 and the preceding remark.

Corollary 2. *If* $E|X_1| = \infty$ *and* $P\{X_1 < 0\} \neq 0$ *then* $P\{S_n > 0 \text{ i.o.}\} = 0$ *or* 1 *according as* $\sum_1^\infty (1/n)P\{S_n > 0\}$ *converges or diverges, according as* $\int_{0+}^{\infty} (x/\int_0^x F(-y)dy)dF(x)$ *is finite or infinite.*

Proof. Corollary 1 and Spitzer's test [4, p. 415, Theorem 2].

Corollary 3. *Let* $\{S_t\}$, $t \geq 0$, *be a process on* R^1 *with stationary independent increments and*

$$\frac{1}{t}\log Ee^{i\theta S_t} = ib\theta - \frac{\sigma^2}{2}\theta^2 + \int\left(e^{i\theta x} - 1 - \frac{i\theta x}{1+x^2}\right)d\lambda(x).$$

Put $\lambda_-(y) = \lambda\{(-\infty, y)\}$, $y < 0$, *and assume* $\lambda_-(-2a) \neq 0$ *for some* $a > 0$. *Then*

$$\limsup_{t\to\infty}\frac{S_t}{t} = +\infty \quad a.s. \text{ iff} \int_a^\infty \left(x\Big/\int_a^x \lambda_-(-y)\,dy\right)d\lambda(x) = \infty.$$

Proof. Write $S_t = S'_t + S''_t$ (in distribution) where

$$\frac{1}{t}\log Ee^{i\theta S'_t} = ib'\theta - \frac{\sigma^2}{2}\theta^2 + \int_{|x|\leq a}(e^{i\theta x} - 1 - i\theta x)\,d\lambda(x),$$

$$\frac{1}{t}\log Ee^{i\theta S''_t} = \int_{|x|>a}(e^{i\theta x} - 1)\,d\lambda(x).$$

Then $\lim_{t\to\infty}(S'_t/t) = ES'_1$, finite, ($E|S'_1|^r < \infty$ for all $r > 0$) and hence

$$\limsup_{t\to\infty}\frac{S_t}{t} = +\infty \quad \text{a.s. iff } \limsup_{t\to\infty}\frac{S''_t}{t} = +\infty \quad \text{a.s.}$$

Now S''_t is a compound Poisson process: $S''_t = X_1 + \cdots + X_{N_t}$, see [3, p. 504, p. 555 and p. 571] where the i.i.d. random variables $\{X_n\}$ have distribution $P\{X_n \in I\} = \beta^{-1}\lambda\{I \cap [-a,a]^c\}$, $\beta = \lambda\{[-a,a]^c\}$ $(0 < \beta < \infty$ by $\lambda_-(-2a) \neq 0$ and properties of Levy measures) and the Poisson process N_t has rate β. Therefore $\lim_{t\to\infty}(N_t/t) = \beta$ a.s., so

$$\beta^{-1}\limsup_{t\to\infty}\frac{S''_t}{t} = \limsup_{n\to\infty}\frac{X_1 + \cdots + X_n}{n} \quad \text{a.s.}$$

and the conclusion of the corollary follows from Theorem 2(a).

3. **Notes.** (1) Suppose $F(x) \leq 1 - c/x^a$ for $x \geq b > 0$ and $\int_{-\infty}^0 |x|^\beta dF(x) < \infty$ for some $0 < \alpha < \beta < 1$. Then $EX_1^+ = \infty$ and $J_- \leq c_1\int_{-\infty}^0 |x|^\alpha dF(x) < \infty$. Hence $S_n/n \to +\infty$ with probability 1. This example is due to C. Derman and H. Robbins [2].

(2) Suppose $F(x) = L(|x|)/|x|^\alpha$, $x \leq -a \leq 0$ where L is slowly varying at ∞ and $0 < \alpha < 1$. Then by Karamata's theorem on regularly varying functions, see [4, p. 281], we have

$$EX_1^- \geq c\int_a^\infty \frac{L(x)}{x^\alpha}dx = \infty$$

and

$$x/m_-(x) \doteq x\Big/\int_a^x y^{-\alpha}L(y)\,dy \sim \frac{(1-\alpha)x^\alpha}{L(x)} = \frac{1-\alpha}{F(-x)}$$

as $x \to \infty$. Hence by Corollary 1

$$P\{\lim S_n = -\infty\} = P\{\lim(S_n/n) = -\infty\} = 1 \tag{3.1}$$

if and only if

$$E(1/F(-X_1^+)) < \infty. \tag{3.2}$$

This example is due to Williamson [7, part (i) of Theorem on p. 866].

In that same paper Williamson conjectured that for arbitrary F (3.2) is necessary and sufficient for (3.1). Here is a counterexample: Let F have a density $F'(x) = f(x)$ such that

$$f(x) \sim \frac{1}{x^2 \log x}, \qquad f(-x) \sim \frac{1}{x^2(\log x)^{1/2}}, \qquad x \to \infty.$$

Then $1 - F(x) \sim (x \log x)^{-1}$, $m_+(x) \sim \log\log x$, $F(-x) \sim x^{-1}(\log x)^{-1/2}$ and $m_-(x) \sim 2(\log x)^{1/2}$ as $x \to \infty$. Hence $J_+ < \infty$ and $J_- = \infty$ and (3.1) holds. But (3.2) fails since $E(1/F(-X_1^+)) \doteq \int_a^\infty x^{-1}(\log x)^{-1/2}dx = \infty$.

(3) If the tails of F satisfy

$$0 < c_1 \le (1 - F(t))/F(-t) \le c_2 < \infty, \qquad t \ge 0, \tag{3.3}$$

then an integration by parts shows that J_+ and J_- both diverge or converge together. Hence the random walk $\{S_n\}$ generated by an F satisfying (3.3) and $E|X_1| = \infty$ is always of the oscillating type; case (iii) of Theorem 1, whether or not it is transient.

(4) Suppose $F'(-x) \sim x^{-2}\log\log x$ and $F'(x) \sim x^{-2}$, $x \to \infty$. Here the left tail predominates: $1 - F(x) = o(F(-x))$ as $x \to \infty$; nevertheless, $\lim\sup(S_n/n) = +\infty$ and $\lim\inf(S_n/n) = -\infty$ with probability 1, since $m_+(x) \sim \log x$, and $m_-(x) \sim \log x \log\log x$ as $x \to \infty$, so for some $a > 0$,

$$J_+ \ge \lim_{t\to\infty}\int_a^t \frac{dx}{x \log x \log\log x} = \lim_{t\to\infty} \log\log\log x\Big]_a^t = \infty,$$

$$J_- \ge \lim_{t\to\infty}\int_a^t \frac{\log\log x}{x \log x}dx = \infty.$$

One should note that the random walk $\{s_n\}$ of this example is transient, i.e. $\lim|S_n| = \infty$ a.s. This follows from the asymptotic estimates $|1 - \varphi(\theta)| \sim |\theta| m_+(1/|\theta|)$, $\mathrm{Re}(1 - \varphi(\theta)) = O(|1 - \varphi(\theta)|/\log(1/|\theta|))$ as $\theta \to 0$ where $\varphi(\theta) = Ee^{iX\theta}$. See [3, Lemma 1].

(5) Theorem 1 guarantees that $\limsup |S_n/n| = \infty$ with probability 1 whenever $EX_1^+ = EX_1^- = \infty$. However, it need *not* happen that

$$P\{\liminf |S_n/n| = \infty\} = 1. \tag{3.4}$$

In fact, given any nonnegative number c there is a random walk $\{S_n\}$ with $EX_1^\pm = \infty$ such that

$$P\{\limsup |S_n/n| = \infty \text{ and } \liminf |S_n/n| = c\} = 1.$$

For the proof see [5, Theorem 7, p. 1196].

Problem. Find a simple integral test equivalent to (3.4). In this connection note Remark 2, p. 1182 in [5].

(6) Put $\varphi(\theta) = Ee^{iX\theta}$. The following assertions are equivalent (see Binmore-Katz [1], also [5, Theorem 6 and Remark 5, p. 1195]:

$$\lim(S_n/n) = +\infty \text{ a.s.} \tag{3.5}$$

$$\lim_{b\to\infty} \int_{-1}^{1} \frac{e^{i\theta b} - 1}{i\theta} \log\left\{1 - \frac{e^{-i\theta a}\varphi(\theta)}{1+\theta^2}\right\}^{-1} d\theta < \infty, \tag{3.6}$$

for every $a > 0$;

$$\sum_{n=1}^{\infty} \frac{1}{n} P\{S_n \le an\} < \infty, \quad \text{for every } a > 0. \tag{3.7}$$

(The convergence of this series, for one a, is of course, Spitzer's criterion for $P\{S_n - an \le 0 \text{ i.o.}\} = 0$.) Thus (3.5)–(3.7) are each equivalent to $J_- < \infty$, $J_+ = \infty$.

Problem. Find a "nonprobabilistic" proof that (3.6) is equivalent to $J_- < \infty$, $J_+ = \infty$.

(7) As noted previously the assertions

$$J_+ + J_- < \infty \tag{3.8}$$

and

$$E|X_1| < \infty \tag{3.9}$$

are equivalent due to Theorem 2 and (2.1). Here is another proof that (3.8) $\Rightarrow$ (3.9).

Proposition. *Let H be a distribution on $[0, \infty)$ with $H(0) < 1$ and put $m(x) = \int_0^x [1 - H(y)]\,dy$; then*

$$I(H) \equiv \int_0^\infty \frac{x}{m(x)} dH(x) < \infty \Leftrightarrow \int_0^\infty x\, dH(x) < \infty.$$

Proof that (3.8) $\Rightarrow$ (3.9) from the Proposition. Let $H(x) = P(|X_1| \leq x) = F(x) - F(-x-)$, $x > 0$, then

$$m(x) = \int_0^x [1 - F(y) + F(-y)]\,dy = m_+(x) + m_-(x)$$

and

$$\begin{aligned} J_+ + J_- &= \int_0^\infty \frac{x}{m_-(x)}\,dF(x) + \int_{-\infty}^0 \frac{|x|}{m_+(|x|)}\,dF(x) \\ &\geq \int_0^\infty \frac{x}{m(x)}\,dH(x) = I(H). \end{aligned}$$

Consequently, $J_+ + J_- < \infty \Rightarrow I(H) < \infty \Rightarrow \int_0^\infty x\,dH(x) = \int_{-\infty}^\infty |x|\,dF(x)$ is infinite.

Proof of the Proposition. The implication $\int_0^\infty x\,dH(x) < \infty \Rightarrow I(H) < \infty$ is clear so let us assume $I(H) < \infty$. Note first that m is absolutely continuous on bounded intervals and

$$m'(x) = 1 - H(x) \leq m(x)/x \quad \text{a.e. } x > 0$$

(the exceptional set where m' does not exist is at most countable); consequently the function $x \to x/m(x)$, $x > 0$, is absolutely continuous on intervals $[a, b]$, $0 < a < b < \infty$ and is nondecreasing because

$$[x/m(x)]' = \frac{m(x) - x[1 - H(x)]}{m^2(x)} \geq 0 \quad \text{a.e.} \tag{3.10}$$

Since $I(H) < \infty$ we see that

$$\epsilon(t) \equiv \frac{t}{m(t)}[1 - H(t)] \leq \int_t^\infty \frac{x}{m(x)}\,dH(x) \to 0 \quad \text{as } t \to \infty$$

and it follows on integrating by parts in $\int_b^\infty (x/m(x))\,dH(x)$ that

$$\int_b^\infty [1 - H(x)]\,d(x/m(x))$$

is finite for any $b > 0$. Choosing $b > 0$ so large that $1 - \epsilon(x) \geq \frac{1}{2}$ for $x \geq b$ and noting (3.10) and the absolute continuity of $\log m(x)$ on bounded intervals $[b, B]$, $b > 0$, gives

$$\begin{aligned} \lim_{t\to\infty} \log \frac{m(t)}{m(b)} &= \int_b^\infty \frac{m'(x)}{m(x)}\,dx \leq 2\int_b^\infty \frac{m'(x)}{m(x)}[1 - \epsilon(x)]\,dx \\ &= 2\int_b^\infty [1 - H(x)]\,d\left(\frac{x}{m(x)}\right) < \infty. \end{aligned}$$

But this implies $\lim_{t\to\infty} m(t) = \int_0^\infty [1 - H(x)]\,dx < \infty$ which in turn implies $\int_0^\infty x\,dH(x) < \infty$.

Note. One can also prove the above proposition by observing $I(H) < \infty$ $\Rightarrow \int_0^x y\,dH(y) \sim m(x)$ as $x \to \infty$, hence

$$\int_0^\infty x\Big/\left(\int_0^x y\,dH(y)\right)\,dH(x) < \infty$$

and then $\int_0^\infty x\,dH(x) < \infty$ by the Abel-Dini theorem.

4. **Proof of Theorem 2.** We prove Theorem 2 in a series of lemmas, each having independent interest.

Lemma 1. *Let G be any probability distribution concentrated on $[0, \infty)$ (but not all the mass at the origin). Put*

$$U(t) = \sum_{n=0}^{\infty} G^{n*}(t), \qquad m(t) = \int_0^t [1 - G(x)]\,dx$$

where G^{n*} is the n-fold convolution. Then

$$1 \le m(t)U(t)/t \le 2 \quad \textit{for all } t > 0 \tag{4.1}$$

and

$$\min(1, a/2) \le U(at)/U(t) \le \max(1, 2a) \tag{4.2}$$

for all $t > 0$, $a > 0$.

Proof. U satisfies the renewal equation $U = 1 + G * U$, see [4, p. 186] or, equivalently,

$$1 = \int_0^x [1 - G(x - y)]\,dU(y)(^2), \qquad x \ge 0.$$

Integrating this over $0 \le x \le t$ gives

$$t = \int_0^t dU(y) \int_y^t [1 - G(x - y)]\,dx = \int_0^t m(t - y)\,dU(y).$$

Since m is nondecreasing

$$m\left(\frac{t}{2}\right)U\left(\frac{t}{2}\right) \le \int_0^{t/2} m(t - y)\,dU(y) \le t \le m(t)U(t)$$

and (4.1) follows. To get (4.2) note that m and U are nondecreasing so

$$1 \le \frac{U(at)}{U(t)} \le \frac{2at}{m(at)U(t)} \le \frac{2at}{m(t)U(t)} \le 2a$$

for $a \ge 1$, $t \ge 0$. Similarly, $U(at)/U(t) \ge a/2$ for $a \le 1$.

(2) Intervals of integration are closed unless otherwise indicated.

Corollary. *An integral of the form* $\int_0^\infty \sum_{n=0}^\infty G^{n*}(ax)\,dF(x)$ *either converges for all* $a > 0$ *or diverges for all* $a > 0$, *according as* $\int_{0+}^\infty (x/m(x))\,dF(x)$ *converges or diverges,* $m(x) = \int_0^x [1 - G(y)]\,dy$.

For Lemmas 2–5 let $\{X_n\}$ be a sequence of i.i.d. random variables with distribution F such that $F(0-) = P\{X_1 < 0\} \neq 0$.

Lemma 2. *Let* $a > 0$ *be fixed and put* $A_0 = \Omega =$ *certain event,* $A_1 = \{X_1 > 0\}$ *and* $A_n = \{X_n^- + \cdots + X_{n-1}^- < aX_n^+\}$, $n > 1$.

(i) *If* $\sum_{n=0}^\infty P(A_n) < \infty$ *then*

$$\limsup (X_n^+/(X_1^- + \cdots + X_n^-)) \leq \frac{1}{a} \quad a.s.$$

(ii) *If* $\sum_{n=0}^\infty P(A_n) = \infty$ *then*

$$\limsup (X_n^+/(X_1^- + \cdots + X_n^-)) \geq \frac{1}{a} \quad a.s.$$

(*We define* $X_n^+(\omega)/0 = \infty$ *if* $X_n(\omega) > 0$.)

Proof. Assertion (i) follows from the first Borel-Cantelli lemma. To prove (ii) assume $\sum P(A_n) = \infty$. Since $P(A_n$ i.o.) is either 0 or 1 by the Hewitt-Savage 0-1 law, it suffices to show

$$P(A_n \text{ i.o.}) > 0. \tag{4.3}$$

Now for $m > n$, $A_n \cap A_m \subset A_n \cap \{X_{n+1}^- + \cdots + X_m^- < aX_m^+\}$ so

$$P(A_n \cap A_m) \leq P(A_n)P(A_{m-n}) \tag{4.4}$$

by independence and stationarity of $\{X_n\}$. Put $Z_n = \sum_{k=0}^n I_{A_k} =$ number of A_k which occur up to time n. Then (4.4) gives

$$EZ_n^2 \leq 2\sum_{i=0}^n P(A_i)\sum_{j=i}^n P(A_{j-i}) \leq 2\left[\sum_{i=0}^n P(A_i)\right]^2 = 2(EZ_n)^2$$

and hence

$$P\{\limsup (Z_n/EZ_n) \geq 1\} > 0$$

by the generalized Borel-Cantelli lemma, cf. [6]. But this clearly implies (4.3) since $EZ_n = \sum_0^n P(A_k) \to \infty$.

Lemma 3. $\limsup (X_n^+/(X_1^- + \cdots + X_n^-)) = 0$ *or* ∞ *with probability* 1, *according as* $J_+ = \int_{0+}^\infty x/m_-(x)\,dF(x)$ *is finite or infinite where* $m_-(x) = \int_0^x F(-y)\,dy$.

Proof. Let A_n be as in Lemma 2. Then since $X_n^- = 0$ on A_n we have

$$\begin{aligned} P\{X_1^- + \cdots + X_n^- < aX_n^+\} &= P(A_n) \\ &= \int_0^\infty P\{X_1^- + \cdots + X_{n-1}^- < ay\} P\{X_n^+ \in dy\} \\ &= \int_{0+}^\infty G^{(n-1)*}(ay -) \, dF(y) \\ &\le \int_{0+}^\infty G^{(n-1)*}(ay) \, dF(y) \end{aligned}$$

where $G(t) = P(X_1^- \le t)$, $t \ge 0$. If $0 < b < a$ then clearly

$$P(A_n) \ge \int_0^\infty G^{(n-1)*}(by) \, dF(y).$$

Therefore from the corollary to Lemma 1 $\sum_1^\infty P\{X_1^- + \cdots + X_n^- < aX_n^+\}$ converges or diverges for all $a > 0$ according as J_+ is finite or infinite. The desired conclusion now follows immediately from Lemma 2.

Lemma 4. *If*

$$\limsup (X_n^+ / (X_1^- + \cdots + X_n^-)) = \infty \quad a.s., \tag{4.5}$$

then $EX_1^+ = \infty$ *and* $\limsup (S_n/n) = \infty$ *a.s., where* $S_n = X_1 + \cdots + X_n$.

Proof. Equation (4.5) implies that the event $X_n^+ \ge 2(X_1^- + \cdots + X_{n-1}^-)$ takes place with probability 1 for infinitely many n. For such an n we have

$$\begin{aligned} S_n &= X_n^+ - (X_1^- + \cdots + X_n^-) + X_1^+ + \cdots + X_{n-1}^+ \\ &\ge |X_1| + |X_2| + \cdots + |X_{n-1}|. \end{aligned}$$

Hence, $S_n/n \ge (|X_1| + \cdots + |X_{n-1}|)/n$ infinitely often with probability 1. However, this implies

$$\limsup \frac{S_n}{n} \ge \liminf \frac{|X_1| + \cdots + |X_n|}{n} \quad \text{a.s.} \tag{4.6}$$

But

$$\lim \frac{|X_1| + \cdots + |X_n|}{n} = E|X_1| \quad \text{a.s.} \tag{4.7}$$

(whether or not $E|X_1|$ is finite), and

$$EX_1^+ = \lim \frac{X_1^+ + \cdots + X_n^+}{n} \ge \limsup \frac{S_n}{n} \quad \text{a.s.} \tag{4.8}$$

since $X_1^+ + \cdots + X_n^+ \ge X_1 + \cdots + X_n = S_n$. It follows from (4.6)–(4.8) that $EX_1^+ \ge E|X_1|$ which, since we are assuming $P(X_1 < 0) > 0$, is impossible unless $EX_1^+ = E|X_1| = \infty$. From (4.6) and (4.7) it now follows that $\limsup(S_n/n) = \infty$ with probability 1.

Lemma 5. *If* $EX_1^+ = \infty$ *and if* $P\{S_n > 0 \text{ i.o.}\} > 0$, *then*

$$\limsup(X_n^+/(X_1^- + \cdots + X_n^-)) = \infty \quad \textit{with probability } 1.$$

This remarkable fact is due to Kesten [5, Theorem 5, p. 1190]. We omit the proof.

Proof of Theorem 2. Note first that we may assume

$$P\{X_1 < 0\} \cdot P\{X_1 > 0\} \neq 0.$$

(If, for example, $P\{X_1 \geq 0\} = 1$, $P\{X_1 = 0\} \neq 1$ then $EX_1 = EX_1^+ < \infty$ if and only if $J_+ < \infty$; see note 7, §3, and Theorem 2 follows from (1.1).)

Clearly the theorem is symmetric in + and − (replace X_n by $\hat{X}_n = -X_n$, then J_+ becomes $\hat{J}_-$, etc.). Thus, (b) follows from (a) and (d) follows from (c).

Proof of (a). If $J_+ = \infty$ then, by Lemmas 3 and 4, $P\{\limsup(S_n/n) = +\infty\} = 1$. Suppose that $P\{\limsup(S_n/n) = +\infty\} = 1$. Then $EX_1^+ = \infty$ (for otherwise by (1.1) we would have $\lim(S_n/n) = EX_1^+ - EX_1^- \neq +\infty$), and obviously $P\{S_n > 0 \text{ i.o.}\} = 1$. Hence, by Lemmas 5 and 3, $J_+ = \infty$.

Proof of (c). Assume $J_+ = \infty$ and $J_- < \infty$. We want to show

$$P\{\lim(S_n/n) = +\infty\} = 1. \tag{4.9}$$

By parts (a) and (b) we have

$$P\{\limsup(S_n/n) = +\infty \text{ and } \liminf(S_n/n) > -\infty\} = 1. \tag{4.10}$$

Also, $EX_1^+ = \infty$ by (2.1). If $EX_1^- < \infty$ then (4.9) follows from (1.1). If, however, $EX_1^- = \infty$ then (4.9) follows from Theorem 1; we must be in case (i) by (4.10). The converse that (4.9) implies $J_+ = \infty$ and $J_- < \infty$ follows from parts (a) and (b) since (4.9) implies

$$P\{\limsup(S_n/n) = \liminf(S_n/n) = +\infty \neq -\infty\} = 1.$$

Added in proof. I have recently learned of a paper *A note on fluctuations of random walks without the first moment* by Tashio Mori, Yokohama Math. J. **20** (1972), 51–55. He has obtained, independently, an integral criterion for $P(S_n > 0 \text{ i.o.}) = 1$ when $E|X_1| = \infty$. His criterion is not expressed in terms of the tails of F, however. Mr. Mori's remark in §1 of his paper that Williamson's result is not true without regular variation of F^- is somewhat misleading: Williamson's result is false even if the tails are regularly varying, (with exponent 1), see Note 2 in §3 above.

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